MAA

MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

32nd Annual

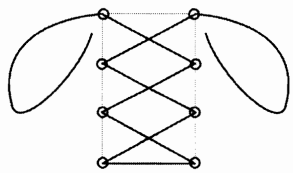
AIME I

American Invitational Mathematics Examination I

Thursday, March 13, 2014

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.
2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.
3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**
4. A combination of the AIME and the American Mathematics Contest 12 are used to determine eligibility for participation in the USA Mathematical Olympiad (USAMO). A combination of the AIME and the American Mathematics Contest 10 are used to determine eligibility for participation in the USA Junior Mathematical Olympiad (USAJMO). The USAMO & the USAJMO will be given in your school on TUESDAY and WEDNESDAY, April 29 & 30, 2014.
5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1. The 8 eyelets for the lace of a sneaker all lie on a rectangle, four equally spaced on each of the longer sides. The rectangle has a width of 50 mm and a length of 80 mm. There is one eyelet at each vertex of the rectangle. The lace itself must pass between the vertex eyelets along a width side of the rectangle and then crisscross between successive eyelets until it reaches the two eyelets at the other width side of the rectangle as shown. After passing through these final eyelets, each of the ends of the lace must extend at least 200 mm farther to allow a knot to be tied. Find the minimum length of the lace in millimeters.



2. An urn contains 4 green balls and 6 blue balls. A second urn contains 16 green balls and N blue balls. A single ball is drawn at random from each urn. The probability that both balls are of the same color is 0.58. Find N.

3. Find the number of rational numbers r, $0 < r < 1$, such that when r is written as a fraction in lowest terms, the numerator and denominator have a sum of 1000.

4. Jon and Steve ride their bicycles along a path that parallels two side-by-side train tracks running in the east/west direction. Jon rides east at 20 miles per hour, and Steve rides west at 20 miles per hour. Two trains of equal length, traveling in opposite directions at constant but different speeds, each pass the two riders. Each train takes exactly 1 minute to go past Jon. The westbound train takes 10 times as long as the eastbound train to go past Steve. The length of each train is $\frac{m}{n}$ miles, where m and n are relatively prime positive integers. Find $m + n$.

5. Let the set $S = \{P_1, P_2, \dots, P_{12}\}$ consist of the twelve vertices of a regular 12-gon. A subset Q of S is called communal if there is a circle such that all points of Q are inside the circle, and all points of S not in Q are outside of the circle. How many communal subsets are there? (Note that the empty set is a communal subset.)

6. The graphs of $y = 3(x - h)^2 + j$ and $y = 2(x - h)^2 + k$ have y-intercepts of 2013 and 2014, respectively, and each graph has two positive integer x-intercepts. Find h.

7. Let w and z be complex numbers such that $|w| = 1$ and $|z| = 10$. Let $\theta = \arg(\frac{w-z}{z})$. The maximum possible value of $\tan^2 \theta$ can be written as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$. (Note that $\arg(w)$, for $w \neq 0$, denotes the measure of the angle that the ray from 0 to w makes with the positive real axis in the complex plane.)

8. The positive integers N and N^2 both end in the same sequence of four digits $abcd$ when written in base 10, where digit a is not zero. Find the three-digit number abc.

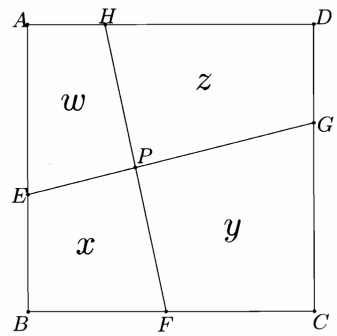
9. Let $x_1 < x_2 < x_3$ be the three real roots of equation $\sqrt{2014}x^3 - 4029x^2 + 2 = 0$. Find $x_2(x_1 + x_3)$.

10. A disk with radius 1 is externally tangent to a disk with radius 5. Let A be the point where the disks are tangent, C be the center of the smaller disk, and E be the center of the larger disk. While the larger disk remains fixed, the smaller disk is allowed to roll along the outside of the larger disk until the smaller disk has turned through an angle of $360°$. That is, if the center of the smaller disk has moved to the point D, and the point on the smaller disk that began at A has now moved to point B, then $\overline{AC}$ is parallel to $\overline{BD}$. Then $\sin^2(\angle BEA) = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

11. A token starts at the point $(0, 0)$ of an xy-coordinate grid and then makes a sequence of six moves. Each move is 1 unit in a direction parallel to one of the coordinate axes. Each move is selected randomly from the four possible directions and independently of the other moves. The probability the token ends at a point on the graph of $|y| = |x|$ is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

12. Let $A = \{1, 2, 3, 4\}$, and let f and g be randomly chosen (not necessarily distinct) functions from A to A. The probability that the range of f and the range of g are disjoint is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m.

13. On square $ABCD$, points E, F, G, and H lie on sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$, respectively, so that $\overline{EG} \perp \overline{FH}$ and $EG = FH = 34$. Segments $\overline{EG}$ and $\overline{FH}$ intersect at a point P, and the areas of quadrilaterals $AEPH$, $BFPE$, $CGPF$, and $DHPG$ are in the ratio 269:275:405:411. Find the area of square $ABCD$.



$w : x : y : z = 269 : 275 : 405 : 411$

14. Let m be the largest real solution to the equation

$$\frac{3}{x-3} + \frac{5}{x-5} + \frac{17}{x-17} + \frac{19}{x-19} = x^2 - 11x - 4.$$

There are positive integers a, b, and c such that $m = a + \sqrt{b + \sqrt{c}}$. Find $a+b+c$.

15. In $\triangle ABC$, $AB = 3$, $BC = 4$, $CA = 5$. Circle ω intersects $\overline{AB}$ at E and B, $\overline{BC}$ at B and D, and $\overline{AC}$ at F and G. Given that $EF = DF$ and $\frac{DG}{EG} = \frac{3}{4}$, length $DE = \frac{a\sqrt{b}}{c}$, where a and c are relatively prime positive integers, and b is a positive integer not divisible by the square of any prime. Find $a + b + c$.

Your Exam Manager will receive a copy of the 2014 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of our publications should be addressed to:*

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2014 USA(J)MO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) and the USA MATHEMATICAL JUNIOR OLYMPIAD (USAJMO) are each a 6-question, 9-hour, essay-type examination. The USA(J)MO will be held in your school on Tuesday and Wednesday, April 29 & 30, 2014. Your teacher has more details on who qualifies for the USA(J)MO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USA(J)MO is to study previous years of these exams. Copies may be ordered from the web site indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the MAA Bookstore or Competitions site at maa.org.